

Constraints on AdS_5 Embeddings

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Abstract

We show that the embedding of either a static or a time dependent maximally 3-symmetric brane with non-zero spatial curvature k into a non-compactified AdS_5 bulk does not yield exponential suppression of the geometry away from the brane. Implications of this result for brane-localized gravity are discussed.

I. STATIC BRANES

It was recently pointed out [1,2] that if our 4-dimensional universe is a 3-brane embedded in an infinite, non-compactified, 5-dimensional bulk AdS_5 spacetime, the AdS_5 bulk geometry could then lead to an exponential suppression of the geometry in distance w away from the brane and thereby localize gravity to it, thus making it possible for us to actually be living in a space with an infinite extra dimension. Specifically, Randall and Sundrum (RS) considered the case of a static Minkowski brane at $w = 0$ with a cosmological constant λ embedded in a non-compactified $w \rightarrow -w$ Z_2 invariant bulk with cosmological constant Λ_5 , and showed that the 5-dimensional Einstein equations

$$G_{AB} = -\kappa_5^2[\Lambda_5 g_{AB} + T_{\mu\nu} \delta_A^\mu \delta_B^\nu \delta(w)] \quad (1)$$

(here $A, B = 0, 1, 2, 3, 5$; $\mu, \nu = 0, 1, 2, 3$, $T_\nu^\mu = \text{diag}(-\lambda, \lambda, \lambda, \lambda)$) had solution

$$ds^2(RS) = dw^2 + e^{-2\xi|w|/R}(-dt^2 + dr^2 + r^2 d\Omega) \quad (2)$$

where $2\xi/R = \kappa_5^2 \lambda/3$, $4\xi^2/R^2 = -2\kappa_5^2 \Lambda_5/3$. A positive sign for λ then leads to an exponential (warp factor) suppression in distance w away from either side of the brane of both the geometry [3] and the graviton propagator [1,2,4], [5] with the propagator for widely separated points on the brane being found to be [1,2,4] the standard 4-dimensional $1/q^2$ one. Intriguing as this possibility is, it is important to see just how general it actually is and just

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how sensitive it is to the matter content on the brane, and, indeed, already even in the RS model itself, no exponential suppression is found in the case where λ is negative (a growing exponential "antiwarp" case). Consequently, in this paper we shall explore a different structure on the brane, one with a more general perfect fluid brane energy-momentum tensor $T^\mu_\nu = \text{diag}(-\rho_b, p_b, p_b, p_b)$ and a geometry [viz. Robertson-Walker (RW)] lower than the maximally 4-symmetric Minkowski one considered by Randall and Sundrum, to then find in the illustrative case of a brane with non-zero constant spatial 3-curvature k embedded in an infinite 5-dimensional spacetime endowed with a cosmological constant, that there is no exponential suppression of the propagator no matter what the signs of ρ_b or k .^[6] Exponential suppression is thus seen not to be a generic property of AdS_5 embeddings.

For the case first of a static, maximally symmetric 3-space embedded in a general (i.e. not yet AdS_5) 5 space,^[7] the most general allowed time independent metric is given by

$$ds^2 = dw^2 - dt^2 e^2(w)/f(w) + f(w)[dr^2/(1 - kr^2) + r^2 d\Omega] \quad (3)$$

In such a geometry the Einstein equations take the form

$$\begin{aligned} G^0_0 &= -3f''/2f + 3k/f = \kappa_5^2[\Lambda_5 + \rho_b \delta(w)], \\ G^1_1 = G^2_2 = G^3_3 &= -f''/2f - e''/e + k/f = -\kappa_5^2[-\Lambda_5 + p_b \delta(w)], \\ G^5_5 &= -3f'e'/2fe + 3k/f = \kappa_5^2\Lambda_5, \end{aligned} \quad (4)$$

with the most general solution being found to be given by (without loss of generality the parameter ν is taken to be positive)

$$\begin{aligned} f(w) &= \alpha e^{\nu|w|} + \beta e^{-\nu|w|} - 2k/\nu^2, \quad e(w) = \alpha e^{\nu|w|} - \beta e^{-\nu|w|}, \\ e^2(w)/f(w) &= f(w) + 4k/\nu^2 - 4(\alpha\beta - k^2/\nu^4)/f(w), \quad \nu = +(-2\kappa_5^2\Lambda_5/3)^{1/2}, \end{aligned} \quad (5)$$

with the Israel junction conditions [8] at the brane^[9] entailing

$$3\nu(\beta - \alpha) = (\alpha + \beta - 2k/\nu^2)\kappa_5^2\rho_b, \quad 6\nu(\alpha + \beta) = (\alpha - \beta)\kappa_5^2(\rho_b + 3p_b). \quad (6)$$

In general then we see [10] that we get both the warp and antiwarp factors (rather than just either one of them) in the RW case, with gravity thus not automatically now being suppressed away from the brane despite the presence of a bulk cosmological constant. Moreover, even if we make the judicious and very specific choice of matter fields $\kappa_5^2(3p_b + \rho_b)^2 + 24\Lambda_5 = 0$, to thereby eliminate the α (or the β) dependent term in Eq. (5) [10], $f(w)$ will still asymptote to a non-vanishing value of $-2k/\nu^2$, with non-zero k (of either possible sign) explicitly preventing localization of gravity.

Now, in reaching this negative result (we show below that the result continues to hold even if we take the geometry to be non-static as well), it is important to note that in the above solution the bulk geometry is not necessarily the AdS_5 one that it would have been in the absence of the brane. And, indeed, for a given bulk geometry to be maximally 5-symmetric, not only must its Einstein tensor be proportional to the metric tensor (as is the case away from the brane in Eq. (1)), but also its Weyl tensor must vanish identically. However, for the geometry associated with the general metric of Eq. (3) 10 components of the bulk Weyl tensor (the 6 $C_{\mu\nu\mu\nu}$ with $\mu \neq \nu$ and the 4 $C_{\mu 5 \mu 5}$) are found to be non-zero, with all of them being found to be kinematically proportional to

$$C_{12}^{12} = [-2eff'' + 3ef'^2 - 2efk - 3e'ff' + 2f^2e'']/12ef^2. \quad (7)$$

We thus see that in the general solution of Eq. (5) the Weyl tensor is not in fact zero (nor even necessarily asymptotically zero - in the $\alpha = 0$ case for instance, $C_{12}^{12} \rightarrow -\kappa_5^2 \Lambda_5/6 \neq 0$).^[11] Moreover, from the point of view of the matter fields on the brane, there is no apparent reason why the bulk geometry should in fact be AdS_5 once a lower symmetry (only 3-symmetric) brane is introduced. However, since it is topologically possible to embed the $k > 0$ S_3 and $k < 0$ R_3 spaces of constant spatial 3-curvature into the spatial part of the $R_1 \times R_4$ universal covering of AdS_5 , it is possible to find particular values of the metric coefficients α and β for which the Weyl tensor will in fact vanish, with it in fact being found to do so in the solution of Eq. (5) provided

$$\alpha\beta - k^2/\nu^4 = 0, \quad (8)$$

a condition under which the Israel junction conditions then require the matter fields to obey the very specific constraint

$$\kappa_5^2 \rho_b (2\rho_b + 3p_b) = 6\Lambda_5, \quad (9)$$

a constraint which incidentally requires the product $p_b \rho_b$ to expressly be negative. Since there is no obvious reason why the bulk and brane matter fields are obliged to have to be related in this very particular manner^[12] (and even if they were, the non-vanishing of k then requires the presence of both the converging and diverging exponentials), we see that, other than in this very special case, the lowering of the symmetry on the brane entails the lowering of the symmetry in the bulk, and that, regardless of what constraint we may or may not impose on the matter fields, in no case do we obtain exponential suppression.

While there would not appear to be any way to avoid this negative result in general, we note that it might still be possible to do so in the restricted case where the post-embedding bulk is in fact taken to be AdS_5 . Specifically, since we would then have two AdS_5 patches in the bulk in this particular case, each one of them can then be brought to the $ds^2 = dw^2 + e^{-2\xi w/R}(-dt^2 + dr^2 + r^2 d\Omega)$ form, (i.e. to the form

$$ds^2 = dw^2 + e^{-2\xi w/R} e^{\pm\nu\eta} (-\nu^2 d\eta^2/4 + d\chi^2 + \sinh^2 \chi d\Omega) \quad (10)$$

where $r = e^{\pm\nu\eta/2} \sinh \chi$, $t = e^{\pm\nu\eta/2} \cosh \chi$, and thus we need to determine which sign of ξ is then found to ensue for each patch. Thus it could be the case that the presence of the diverging exponential might only have been due to a particular choice of coordinates. As we shall see though, this will not in fact turn out to be the case, though investigation of the issue will prove to be instructive.

In analyzing the $\alpha\beta - k^2/\nu^4 = 0$ case, we note first that since in this case the metric coefficients are given by $f(w) = \alpha e^{\nu|w|} + \beta e^{-\nu|w|} - 2k/\nu^2$, $e^2/f = \alpha e^{\nu|w|} + \beta e^{-\nu|w|} + 2k/\nu^2$, we see that there is no point at which both metric coefficients can simultaneously vanish. However, it is possible for one to vanish somewhere. Specifically, if we define $\alpha = e^{\sigma - \nu w_0}$, $\beta = e^{\sigma + \nu w_0}$, $e^{2\nu w_0} = \beta/\alpha$, $e^\sigma = |k|/\nu^2$, for $k = +1$ we find that

$$f = (4/\nu^2) \sinh^2(\nu w_0/2 - \nu|w|/2), \quad e^2/f = (4/\nu^2) \cosh^2(\nu w_0/2 - \nu|w|/2), \quad (11)$$

and for $k = -1$ that

$$f = (4/\nu^2)\cosh^2(\nu w_0/2 - \nu|w|/2), \quad e^2/f = (4/\nu^2)\sinh^2(\nu w_0/2 - \nu|w|/2). \quad (12)$$

Thus for w_0 negative, all of these metric coefficients will be at their absolute minimum values at the brane, and thus diverge away from the brane no matter what the sign of k , but for w_0 positive, they will all be at local maxima at the brane, and actually fall (monotonically) until the points $w = \pm w_0$ are reached.^[13] Discussion of the various cases thus depends on whether w_0 is positive or negative, i.e. on whether β/α is greater or lesser than one. Since for a Minkowski signed metric the quantity $f(0) = \alpha + \beta - 2k/\nu^2$ must necessarily be positive, we see from the Israel junction conditions of Eq. (6) that w_0 is negative if ρ_b is negative, while being positive if ρ_b is positive. We must thus treat these two cases separately, and since for our purposes here it is sufficient to show that there is at least one case in which there is an $\xi < 0$ antiwarp factor when the RW metric is brought to a form analogous to that of Eq. (2), we shall now investigate this issue in detail in the negative curvature case.

While not at all essential in the following, we nonetheless find it convenient to normalize the fields so that $f(0) = 1$.^[14] On restricting now to $k = -1$, in the $\rho_b < 0$ case first where the parameter w_0 is given by $w_0 = -\nu^{-1}L_+(1)$ where

$$L_{\pm}(f) = \log[\nu^2 f/2 - 1 \pm \nu(\nu^2 f^2 - 4f)^{1/2}/2] = -L_{\mp}(f), \quad (13)$$

the coordinate transformation

$$w = [w_0 + \nu^{-1}L_+(\gamma)]\theta(\gamma - 1) - [w_0 + \nu^{-1}L_+(1/\gamma)]\theta(1 - \gamma), \quad r = \sinh\chi', \\ 2t = \{\nu w' + \log[(\nu^2\gamma - 4)/(\nu^2 - 4)]\}\theta(\gamma - 1) + \{\nu\eta' - \log[(\nu^2 - 4\gamma)/(\nu^2 - 4)]\}\theta(1 - \gamma) \quad (14)$$

where $\gamma = e^{\nu\eta' - \nu w'}$ (and where $\theta(w_0 + \nu^{-1}L_+(\gamma)) = \theta(\gamma - 1)$ for w_0 negative) is found to bring the metric of Eq. (3) to the form

$$ds^2 = dw'^2 + [e^{\nu(\eta' - w')}\theta(\eta' - w') + e^{-\nu(\eta' - w')}\theta(w' - \eta')] \\ \times [-\nu^2 d\eta'^2/4 + d\chi'^2 + \sinh^2\chi' d\Omega]. \quad (15)$$

Comparing with Eq. (10) we thus recognize Eq. (15) to precisely possess an antiwarp factor, just as we had anticipated. Additionally, we also see that in the antiwarp factor coordinate system associated with Eq. (15) the brane is not at rest. Thus once the brane has a matter field source other than a pure cosmological constant, it will not be at rest in a pure warp factor or pure antiwarp factor coordinate system. Moreover, it is to be expected that the brane would not be at rest in Eq. (15) since the original Z_2 symmetry was with respect to the coordinate w , with there being no Z_2 symmetry in the new coordinate w' . (Rather the Z_2 symmetry of the $e^{\nu(\eta' - w')}\theta(\eta' - w') + e^{-\nu(\eta' - w')}\theta(w' - \eta')$ term is with respect to the light cone coordinate $w' - \eta'$ instead.) It is thus the absence of any $w' \rightarrow -w'$ symmetry in the primed coordinate system which prevents Eq. (15) from leading to localization of gravity.^[15]

In the $\rho_b > 0$ case where $w_0 = \nu^{-1}L_+(1)$, the coordinate transformation^[16]

$$w = [w_0 + \nu^{-1}L_+(\gamma)]\theta(w_0 + \nu^{-1}L_+(\gamma)) + [w_0 + \nu^{-1}L_-(\gamma)]\theta(w_0 + \nu^{-1}L_-(\gamma)) \\ - [w_0 + \nu^{-1}L_-(1/\gamma)]\theta(w_0 + \nu^{-1}L_-(1/\gamma)) - [w_0 + \nu^{-1}L_+(1/\gamma)]\theta(w_0 + \nu^{-1}L_+(1/\gamma)), \quad (16)$$

$$2t = \{\nu w' + \log[(\nu^2\gamma - 4)/(\nu^2 - 4)]\}\theta(w_0 + \nu^{-1}L_+(\gamma)) + \\ \{-\nu w' - \log[(\nu^2\gamma - 4)/(\nu^2 - 4)]\}\theta(w_0 + \nu^{-1}L_-(\gamma)) + \\ \{-\nu\eta' + \log[(\nu^2 - 4\gamma)/(\nu^2 - 4)]\}\theta(w_0 + \nu^{-1}L_-(1/\gamma)) + \\ \{\nu\eta' - \log[(\nu^2 - 4\gamma)/(\nu^2 - 4)]\}\theta(w_0 + \nu^{-1}L_+(1/\gamma)) \quad (17)$$

brings the metric to the form

$$ds^2 = dw'^2 + \{e^{\nu(\eta' - w')}\theta(\gamma - 4/\nu^2)[1 + \theta(1 - \gamma)] + e^{-\nu(\eta' - w')}\theta(\nu^2/4 - \gamma)[1 + \theta(\gamma - 1)]\} \times [-\nu^2 d\eta'^2/4 + d\chi'^2 + \sinh^2 \chi' d\Omega]. \quad (18)$$

Thus for ρ_b positive we again generate an antiwarp factor, with gravity thus not being localized to the brane for non-zero k even for "attractive" ρ_b positive matter sources.^[17] Consequently, having an explicit AdS_5 geometry in the bulk is not sufficient in and of itself to always guarantee localization of gravity.

II. NON-STATIC BRANES

For a non-static, maximally symmetric 3-space embedded in a general (i.e. not yet AdS_5) 5 space, the most general allowed metric can be taken to be given by^[18]

$$ds^2 = dw^2 - dt^2 e^2(w, t)/f(w, t) + f(w, t)[dr^2/(1 - kr^2) + r^2 d\Omega]. \quad (19)$$

For this metric the components of the Einstein tensor are given by (the dot and the prime denote derivatives with respect to t and w respectively)

$$\begin{aligned} G^0_0 &= -3f''/2f + 3k/f + 3\dot{f}^2/4e^2f, \\ G^1_1 = G^2_2 = G^3_3 &= -f''/2f - e''/e + k/f + \ddot{f}/e^2 + \dot{f}^2/4e^2f - \dot{e}\dot{f}/e^3, \\ G^5_5 &= -3f'e'/2fe + 3k/f + 3\ddot{f}/2e^2 + 3\dot{f}^2/4e^2f - 3\dot{e}\dot{f}/2e^3, \\ G^5_0 &= 3e'\dot{f}/2e^3 - 3\dot{f}'/2e^2, \end{aligned} \quad (20)$$

while the non-vanishing components of the Weyl tensor (the 6 $C_{\mu\nu\mu\nu}$ with $\mu \neq \nu$ and the 4 $C_{\mu 5 \mu 5}$) are all found to be proportional to

$$C_{0505} = (4e^3 f f'' - 6e^3 f'^2 + 4e^3 f k + 6e^2 f e' f' - 4e^2 f^2 e'' - 2e f^2 \ddot{f} + e f \dot{f}^2 + 2f^2 \dot{e} \dot{f})/8e f^3, \quad (21)$$

with the kinematic relation

$$4f^3 C_{0505} + 2e^2 f^2 (G^5_5 - G^0_0 - G^3_3) = 6e^2 f f'' - 3e^2 f'^2 \quad (22)$$

being found to hold identically (in the coordinate gauge associated with Eq. (19)).

Solving the theory with the same (but now time dependent) sources as in the static case discussed earlier is now straightforward. The non-trivial ($\dot{f} \neq 0$) vanishing of G^5_0 entails that

$$e = A(t)\dot{f} \quad (23)$$

where $A(t)$ is an arbitrary function of t . Setting G^0_0 equal to $-3\nu^2/2$ in the bulk entails that

$$f'' = 1/2A^2 + 2k + \nu^2 f \quad (24)$$

so that

$$\begin{aligned} f &= -1/2A^2\nu^2 - 2k/\nu^2 + \alpha e^{\nu|w|} + \beta e^{-\nu|w|}, \\ e &= \dot{A}/A^2\nu^2 + A\dot{\alpha}e^{\nu|w|} + A\dot{\beta}e^{-\nu|w|}, \end{aligned} \quad (25)$$

where α and β depend on t . Similarly, setting G^5_5 equal to $-3\nu^2/2$ in the bulk entails that

$$f'^2 - 4kf - \nu^2 f^2 - f/A^2 = B(|w|) \quad (26)$$

where $B(|w|)$ is an arbitrary function of $|w|$ which must be continuous at the brane.^[19] However, compatibility with Eq. (24) then entails that B must actually be constant, with the time dependent functions in the solution of Eq. (25) then being related according to^[20]

$$(1 + 4A^2k)^2/16A^4 - \nu^4\alpha\beta = \nu^2 B/4. \quad (27)$$

Further, the Israel junction conditions at the brane, yield (see e.g. Binetruy et. al. [6])

$$3\nu(\beta - \alpha) = f(0, t)\kappa_5^2\rho_b, \quad 6\nu[\dot{\alpha} - \dot{\beta}] = \dot{f}(0, t)\kappa_5^2(\rho_b + 3p_b) \quad (28)$$

(with the standard $\dot{\rho}_b + 3\dot{f}(0, t)(\rho_b + p_b)/2f(0, t) = 0$ then being recovered). Thus, just as in the static case, we once again find that in general (i.e. for completely arbitrary ρ_b, p_b) there is no asymptotic suppression of the metric coefficients.

Moreover, if we additionally require the bulk to be AdS_5 , the requisite vanishing of the Weyl tensor then imposes the additional condition

$$2ff'' - f'^2 - \nu^2 f^2 = 0 \quad (29)$$

in the bulk, to require that B actually be zero, with α and β then having to be related to each other according to

$$(1 + 4A^2k)^2/16A^4 = \nu^4\alpha\beta, \quad (30)$$

with the metric coefficient $f(w, t)$ then being given as

$$f(w, t) = (\alpha^{1/2}e^{\nu|w|/2} - \beta^{1/2}e^{-\nu|w|/2})^2. \quad (31)$$

From Eq. (30) we see that product $\alpha\beta$ must thus be non-negative (and even necessarily greater than zero in the $k > 0$ case). Hence even in the time dependent case, there is still no exponential suppression of the geometry for a non-spatially flat brane embedded in an AdS_5 bulk.

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- [5] With metric fluctuations of the form $ds^2 = dw^2 + dx^\mu dx^\nu (e^{2f(w)}\eta_{\mu\nu} + h_{\mu\nu})$ obeying $(\partial_w^2 - 2f'' - 4f'^2 - e^{-2f}q^\lambda q_\lambda)h_{\mu\nu}(w, q_\lambda) = 0$, localization of the geometry entails localization of gravity ($h_{\mu\nu}(w, q^2 = 0) \simeq e^{2f}$), with the coordinate independent gravitational scattering S-matrix then being highly suppressed when it involves a scattering between a particle which is localized to the brane and one which is not.
- [6] While we concentrate here on localization and embedding issues, the Robertson-Walker brane theory has also been studied as a cosmology in and of itself, by e.g. C. Csaki, M. Graesser, C. Kolda and J. Terning, Phys. Lett. B**462**, 34 (1999), by J. M. Cline, C. Grojean and G. Servant, Phys. Rev. Lett. **83** 4245 (1999), by P. Binetruy, C. Deffayet and D. Langlois, Nucl. Phys. B **565**, 269 (2000), by P. Binetruy, C. Deffayet, U. Ellwanger and D. Langlois, Phys. Lett. B**477**, 285 (2000), and by R. N. Mohapatra, A. Perez-Lorenzana and C. A. de S. Pires, Cosmology of Brane-Bulk Models in Five Dimensions, hep-ph/0003328. The stability and observational viability of such cosmologies was established by C. Csaki, M. Graesser, L. Randall and J. Terning, Phys. Rev. D**62**, 045015 (2000) in the particular case of linearized perturbations around Minkowski (and thus spatially flat) branes due to the addition of weak perfect fluid matter fields on them. The implications for localization of non-zero k branes (with not necessarily weak matter fluids) which we present here and in a companion paper [10], as well as an analysis of the geometry of the bulk (i.e. whether it is or is not AdS_5) in such cases, appear not to have been considered in the previous literature.
- [7] While the geometry of the bulk might be pure AdS_5 in the absence of the brane, once the brane is present it is able to set up a gravitational field of its own in the bulk, to thus potentially lead to a bulk geometry other than AdS_5 despite the fact that the only matter field in the bulk itself is a bulk cosmological constant.
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- [9] These conditions take the generic form $K_{\mu\nu}(w = 0^+) - K_{\mu\nu}(w = 0^-) = -\kappa_5^2(T_{\mu\nu} - q_{\mu\nu}T^\alpha_\alpha/3)$ where $q_{\mu\nu}$ is the induced metric on the brane, with the brane extrinsic curvature having the two independent components $K^4_4 = e'/e - f'/2f$, $K^1_1 = K^2_2 = K^3_3 = f'/2f$ in the RW brane case.

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- [11] This situation is to be contrasted with the Minkowski case brane considered by Randall and Sundrum themselves, since the assumption of a Minkowski brane (and thus of a maximally 4-symmetric, pure cosmological constant perfect fluid on the brane) entails that $e(w) = f(w)$ and $k = 0$, to thus automatically ensure the kinematic vanishing of the bulk Weyl tensor (i.e. the metric $ds^2 = dw^2 + f(w)(-dt^2 + dx^2 + dy^2 + dz^2)$ is kinematically conformal to flat for any $f(w)$), with the Einstein equations then not only obliging the bulk geometry to be AdS_5 but also forcing one of the two α and β coefficients to vanish, to thereby entail only one exponential term.
- [12] Unless an AdS_5 bulk is enforced upon us a priori, say, perhaps, by a dimensional compactification of string theory; with the brane field configuration required for Eq. (9) then possibly being an explicit consequence of the embedding, with the brane pressure, rather than being an a priori input, instead being [10] the output pressure produced on the brane in response to the pressure generated by the gravitational field in the bulk. As regards Eq. (9) we also note in passing that should it hold, it then has an interesting implication for the brane cosmology which then ensues. Specifically, it was noted by T. Shiromizu, K. Maeda and M. Sasaki, Phys. Rev. D **62**, 024012 (2000), that because of the embedding the leading terms in the Einstein equations on the brane have to take the form ${}^{(4)}G_{\mu\nu} = \kappa_5^2(6\Lambda_5 + \kappa_5^2\lambda^2)q_{\mu\nu}/12 - 8\pi G_N[(\rho_m + p_m)U_\mu U_\nu + p_m q_{\mu\nu}]$ where $G_N = \lambda\kappa_5^4/48\pi$ and where we have taken the matter density on the brane to be given as $\rho_b = \lambda + \rho_m$, $p_b = -\lambda + p_m$. In the presence of Eq. (9) the Einstein tensor on the brane is then given by ${}^{(4)}G_{\mu\nu} = -8\pi G_N[(\rho_m + p_m)U_\mu U_\nu + p_m q_{\mu\nu} - (\rho_m + 3p_m)q_{\mu\nu}/2] + O(\kappa_5^4\rho^2)$, with the leading order source acting just like a perfect fluid with energy density $\rho = 3(\rho_m + p_m)/2$ and pressure $p = -(\rho_m + p_m)/2 = -\rho/3$, i.e. just like negative pressure quintessence.
- [13] Moreover, they will even fall exponentially fast in the $w \ll w_0$ region, and thus would actually lead to a localization of gravity if the theory could sensibly be compactified either by an identification of the $\pm w_0$ points or by a dynamical mechanism which would disconnect the $|w| > w_0$ sector of the theory all together. In fact a dynamical mechanism for obtaining such a compactification (one based on a dynamical scalar field) has explicitly been identified (A. Davidson and P. D. Mannheim, Dynamical Localization of Gravity, MIT-CTP-3015, hep-th/0009064) in the case of a dS_4 brane embedded in AdS_5 , a case for which the metric is given by $ds^2 = dw^2 + \sinh^2(w_0 - |w|)[e^{2at}(dx^2 + dy^2 + dz^2) - dt^2]$, with a compactification actually being achievable in the dS_4 case because of the vanishing of the entire 4-dimensional part of the metric at the compactification points. Since such an overall vanishing of the metric coefficients does not occur in the RW case, it remains to be seen whether compactification could in fact lead to localization of gravity in the RW case, and even then, for such a mechanism to be of any possible practical interest, it would need to be shown that the compactification radius could reasonably be taken to be macroscopic. As a possibility though it would appear to merit further study (perhaps being achievable by locating additional branes at the compactification points) since the analysis of this paper shows that localization of gravity for just one solitary brane all on its own cannot be achieved if the AdS_5 space is to be the same infinite extent, non-compactified one considered by Randall and Sundrum in the Minkowski brane case.

- [14] I.e. we pick the fields so that $\alpha = [\nu^2 + 2k \pm \nu(\nu^2 + 4k)^{1/2}]/2\nu^2$, $\beta = [\nu^2 + 2k \mp \nu(\nu^2 + 4k)^{1/2}]/2\nu^2$ (i.e. $\nu^2 + 4k \geq 0$), $\kappa_5^2 \rho_b = \mp 3(\nu^2 + 4k)^{1/2}$, $\kappa_5^2 p_b = \pm (3\nu^2 + 8k)/(\nu^2 + 4k)^{1/2}$, $\kappa_5^4 \rho_b(\rho_b + p_b) = 12k$.
- [15] With regard to the coordinate transformations of Eq. (14), we also note in passing that the $n^A = n^5 \delta_5^A$ normal to the brane (as given in the original Eq. (3) coordinate system) transforms so that it acquires components in both the w' and η' directions, components which then explicitly become dependent on both w' and η' . Consequently, both the induced metric $q_{\mu\nu} dx^\mu dx^\nu$ on the brane and the brane extrinsic curvature $K_{\mu\nu} = q^\sigma{}_\mu q^\tau{}_\nu n_{\sigma;\tau}$ acquire $(0, 5)$ and $(5, 5)$ components in the transformed coordinate system, to thus enable the Israel junction conditions at the brane to be compatible with matter field sources more general than the pure brane cosmological constant source associated with a brane at rest in a warp factor coordinate system. In fact, if the brane were at rest in the Eq. (2) coordinate system, the junction conditions would then require the brane matter fluid to be a pure cosmological constant term. Consequently, once the brane matter fluid departs from a pure cosmological constant, the brane can not be at rest in the warp factor coordinate system, with the brane extrinsic curvature having to then have an expressly time-dependent discontinuity.
- [16] For w_0 positive the range for $\theta(w_0 + \nu^{-1} L_+(\gamma))$ is $\gamma > 4/\nu^2$, and that of $\theta(w_0 + \nu^{-1} L_-(\gamma))$ is $1 > \gamma > 4/\nu^2$.
- [17] With only the flat $k = 0$ Robertson-Walker case thus being localizable by an AdS_5 embedding (a case where the vanishing of C^{12}_{12} entails $e = f$), it would be of interest to see whether geometries with non-zero k but with a negligibly small current era value of $\Omega_k(t) = -kc^2/\dot{R}^2(t)$ could still admit of an effective brane-localized current era gravity, with such localization then having only set in during an early universe inflationary phase.
- [18] Under the two Hamilton-Jacobi equation type coordinate transformations $g^{ww} = g^{\mu\nu}(\partial w/\partial x_\mu)(\partial w/\partial x_\nu) = 1$, $g^{tw} = g^{\mu\nu}(\partial t/\partial x_\mu)(\partial w/\partial x_\nu) = 0$ in the two dimensional (x^1, x^2) space, the general maximally 3-symmetric $ds^2 = g_{11}(dx^1)^2 + 2g_{12}dx^1 dx^2 + g_{22}(dx^2)^2 + f(x^1, x^2)[dr^2/(1 - kr^2) + r^2 d\Omega]$ metric can be brought to the form exhibited in Eq. (19). Consequently, the metric coefficient g_{ww} will only depart from one in fluctuations of the geometry which are less than maximally 3-symmetric.
- [19] While $B(|w|)$ could in principle take different values on the two sides of the brane, because it is related to f'^2 it in fact does not.
- [20] Given Eq. (27), the one remaining Einstein equation (which can conveniently be given as $G^5_5 - G^0_0/2 - 3G^3_3/2 = 3(e f'' + f e'' - e' f')/2ef = 3\nu^2/2$ in the bulk) is then satisfied identically by the solution of Eq. (25).